

Noncommutative circle bundles and new Dirac operators

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Abstract

We study spectral triples over noncommutative principal $U(1)$ bundles. Basing on the classical situation and the abstract algebraic approach, we propose an operatorial definition for a connection and compatibility between the connection and the Dirac operator on the total space and on the base space of the bundle. We analyze in details the example of the noncommutative three-torus viewed as a $U(1)$ bundle over the noncommutative two-torus and find all connections compatible with an admissible Dirac operator. Conversely, we find a family of new Dirac operators on the noncommutative tori, which arise from the base-space Dirac operator and a suitable connection.

1 Introduction

The principal $U(1)$ bundles are the simplest and most fundamental examples of fibre bundles, often encountered in mathematics and physics. They are usually equipped both with a connection and a metric, which are in principle independent of each other, though, an interesting situation arises when they are compatible in

some natural way. This is reflected, for example, in the spectral geometry of $U(1)$ bundles, which in terms of Laplace operator has been studied in [10], whereas the analysis of Dirac operator was presented in [1, 2]. In this note we shall extend part of the latter analysis to the analogue of principal $U(1)$ bundles in noncommutative geometry, encoding their geometric aspects in terms of spectral triples [5, 6].

2 Spin Geometry of $U(1)$ -bundles

We suppose that M is a $n + 1$ dimensional ($n + 1$ odd) compact manifold which is the total space of $U(1)$ -principal bundle over the n -dimensional (n even) base space $N = M/U(1)$. Moreover assume M is equipped with a Riemannian metric \tilde{g} and the $U(1)$ action (free and transitive on fibres) is isometric. The base space N carries a unique metric g such that the projection $\pi : (M, \tilde{g}) \rightarrow (N, g)$ is a Riemannian submersion.

We can and shall use a suitable local orthonormal frame (basis) of the tangent space TM , $e = (e_0, e_1, \dots, e_n)$, such that e is $U(1)$ invariant and e_0 is the (normalized) Killing vector field K associated to the $U(1)$ -action. For simplicity we assume that the fibres are of constant length $2\pi\ell$.

There exists a unique principal connection 1-form $\omega : TM \rightarrow \mathbb{R} \approx u(1)$, such that $\ker \omega$ is orthogonal to the fibres for all $m \in M$ with respect to \tilde{g} . Obviously it is given by $\omega = e^0/\ell$, where (e^0, e^1, \dots, e^n) is the dual frame to e . Conversely, if we are given a principal connection on the principal $U(1)$ bundle and a metric on the base space N then there exists a unique $U(1)$ -invariant metric on M , such that the horizontal vectors are orthogonal to the fundamental (Killing) vector field K of length ℓ .

Assume now that M is spin and let ΣM be its spinor bundle (which is hermitian, rank $2^{\frac{n}{2}}$ complex vector bundle). The $U(1)$ action either lifts to the spin structure and then to an action

$$\kappa : U(1) \times \Sigma M \rightarrow \Sigma M,$$

or to a projective action (up to a sign), i.e. to the action of a non-trivial double cover of $U(1)$, which happens to be still $U(1)$ as a group.

Assuming the former, we have a *projectable* spin structure on M . As explained in [2] this induces a spin structure on N . Conversely, any spin structure on N canonically induces a projectable spin structure on M via a pull-back construction. We recall that the Dirac operator \tilde{D} on M can be constructed as follows. Let γ_j , $j = 0, 1, \dots, n$, be the antihermitian matrices in $M(2^{\frac{n}{2}}, \mathbb{C})$, which satisfy the relations

$$\gamma_j \gamma_k + \gamma_k \gamma_j = -2\delta_{jk}. \quad (2.1)$$

Then Dirac operator \tilde{D} acting on sections of ΣM can be explicitly written as

$$\tilde{D} = \sum_{i=0}^n \gamma_i \partial_{e_i} + \frac{1}{4} \sum_{i,j,k=0}^n \tilde{\Gamma}_{ij}^k \gamma_i \gamma_j \gamma_k,$$

where $\tilde{\Gamma}_{ij}^k$ are Christoffel symbols (in the orthonormal basis e) of the Levi-Civita connection on M . In particular

$$\begin{aligned} -\tilde{\Gamma}_{ij}^0 &= \tilde{\Gamma}_{i0}^j = \tilde{\Gamma}_{0i}^j = \frac{\ell}{2} d\omega(e_i, e_j), \\ \tilde{\Gamma}_{i0}^0 &= \tilde{\Gamma}_{0i}^0 = \tilde{\Gamma}_{00}^i = \tilde{\Gamma}_{00}^0 = 0. \end{aligned} \tag{2.2}$$

Since the metric on M is completely characterized by the connection 1-form ω , the length ℓ of the fibres and the metric g on N , the Dirac operator \tilde{D} on M can be expressed in terms of ω , and g . Conversely, the metric on N , the connection ω and the length of the fibres can be recovered from \tilde{D} .

Following this line, Ammann and Bär showed [2] how to present the Dirac operator \tilde{D} as a sum of two first order differential operators on $L^2(\Sigma M)$ and a zero order term (endomorphism of the spinor bundle).

The first operator, called the *vertical* Dirac operator is

$$D_v := \frac{1}{\ell} \gamma_0 \partial_K,$$

where

$$\partial_K(\Psi)(m) = \frac{d}{dt} \Big|_{t=0} \kappa(e^{-it}, \Psi(m \cdot e^{it}))$$

is the Lie derivative of a spinor Ψ along the $U(1)$ Killing field. Note that $D_v := \gamma_0 \partial_{e_0}$, where ∂_{e_0} could be interpreted as the Dirac operator associated to the typical fibre $S^1 \simeq U(1)$, whereas γ_0 is the Clifford representation of the (normalized) one-form $e^0 = \ell\omega$.

It follows from (2.2) that the spinor covariant derivative differs from the Lie derivative in the direction of e_0 :

$$\nabla_{e_0} = \partial_{e_0} + \frac{\ell}{4} \sum_{j < k} d\omega(e_j, e_k) \gamma_j \gamma_k. \tag{2.3}$$

The description of the second differential operator D_h , called a *horizontal* Dirac operator, uses an orthogonal decomposition of the Hilbert space into irreducible representations of $U(1)$:

$$L^2(\Sigma M) = \bigoplus_{k \in \mathbb{Z}} V_k,$$

where V_k are the closures of eigenspaces of the Lie derivative ∂_{e_0} for the eigenvalue ik , $k \in \mathbb{Z}$. This decomposition is preserved by \tilde{D} , since it commutes with the (isometric) $U(1)$ -action on M .

Next, let $L := M \times_{U(1)} \mathbb{C}$ be the complex line bundle associated to the $U(1)$ -bundle $M \rightarrow N$. In [2] it is shown that there is a natural homothety of Hilbert spaces (isomorphism if the fibres are of length $\ell = 1$)

$$Q_k : L^2(\Sigma N \otimes L^{-k}) \rightarrow V_k,$$

which satisfies

$$Q_k(\gamma_i \Psi) = \gamma_i Q_k(\Psi), \quad i = 1, \dots, n$$

and

$$\nabla_{e_i} Q_k(\Psi) = Q_k(\nabla_{f_i} \Psi) + \frac{1}{4} \sum_{j=1}^n \left(\tilde{\Gamma}_{i0}^j - \tilde{\Gamma}_{ij}^0 \right) \gamma_0 \gamma_j Q_k(\Psi), \quad (2.4)$$

where $f = (f_1, f_2, \dots, f_n)$, $f_i := \pi_*(e_i)$ is a local orthonormal frame on N .

Then $D_h : L^2(\Sigma M) \rightarrow L^2(\Sigma M)$ is defined as the unique closed linear operator, such that on each V_k it is:

$$D_h := Q_k \circ D'_k \circ Q_k^{-1},$$

where $D'_k = \sum_{i=1}^n \gamma_i \otimes \text{id} (\nabla_{f_i}^N \otimes \text{id} + \text{id} \otimes k \nabla_{f_i}^\omega)$ is the twisted (of charge k) Dirac operator on $\Sigma N \otimes L^{-k}$.

Here, ∇^N is the covariant spinor derivative on N coming from the Levi-Civita connection on N , whose Christoffel symbols with respect to the projected frame $f = (f_1, \dots, f_n)$ on N are given by

$$\Gamma_{ij}^k = \tilde{\Gamma}_{ij}^k, \quad \forall i, j, k \in \{1, \dots, n\} \quad (2.5)$$

and $k \nabla^\omega$ is the covariant derivative on L^{-k} of the connection ω . Using the above results the Dirac operator \tilde{D} on M can be expressed as a sum

$$\tilde{D} = D_v + D_h + Z,$$

where

$$Z := -(\ell/4) \gamma_0 \sum_{j < k} d\omega(e_j, e_k) \gamma_j \gamma_k.$$

Observe that since D_h , γ_0 and Z are $U(1)$ -invariant, they commute with ∂_{e_0} . Since for even n , $\gamma_1 \gamma_2 \dots \gamma_n$ anticommutes with any twisted Dirac operator on N and $\gamma_0 \sim \gamma_1 \gamma_2 \dots \gamma_n$ (up to a constant $1, i, -1, -i$ depending on n and the representation of gamma matrices), γ_0 anticommutes with D_h .

Finally, let us observe that the presence of the zero-order term Z is responsible for the torsion-free condition. In other words, omitting Z still provides a Dirac operator of M for the linear connection, which preserves the metric \tilde{g} but has a non-vanishing (in general) torsion. This can be seen easily by looking at the Christoffel symbols defined by (2.5) and by (2.2). If, in the latter formula we put $\tilde{\Gamma}_{ij}^k = 0$ whenever one or more of the indices i, j, k is zero, we get a linear connection, which is still compatible with the metric but the components

$$T_{ij}^0 = e^0(\nabla_{e_i} e_j - \nabla_{e_j} e_i - [e_i, e_j]) = de^0(e_i, e_j) = \ell d\omega(e_i, e_j) \quad (2.6)$$

of the torsion tensor do not vanish (in general).

3 Noncommutative $U(1)$ principal bundles

We turn now to the noncommutative picture, where the concept of principal bundles is given by the Hopf-Galois theory. Let us shortly recall the basic definitions, for details and examples see [3, 4, 8, 12].

Let H be a unital Hopf algebra and \mathcal{A} be a right H -comodule algebra. We will use the natural Sweedler notation for the right coaction of H on \mathcal{A} :

$$\Delta_R(a) = a_{(0)} \otimes a_{(1)} \in \mathcal{A} \otimes H.$$

We denote by \mathcal{B} the subalgebra of coinvariant elements of \mathcal{A} .

Definition 3.1. A $\mathcal{B} \hookrightarrow \mathcal{A}$ is Hopf-Galois extension iff the canonical map $\chi :$

$$\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \ni a' \otimes a \mapsto \chi(a' \otimes a) = a' a_{(0)} \otimes a_{(1)} \in \mathcal{A} \otimes H, \quad (3.1)$$

is an isomorphism.

In the purely algebraic settings the (principal) connections are defined as certain maps from the Hopf algebra H to the first order universal differential calculus $\Omega_u^1(\mathcal{A}) := \ker m_{\mathcal{A}} \subset \mathcal{A} \otimes \mathcal{A}$ with the exterior derivative $d_u : \mathcal{A} \rightarrow \Omega_u^1(\mathcal{A})$, $d_u a := 1 \otimes a - a \otimes 1$.

Let $\text{Ad}_R(h) = h_{(2)} \otimes S(h_{(1)})h_{(3)}$ be the right adjoint coaction and ε the counit of H .

Definition 3.2. A map $\omega : H \rightarrow \Omega_u^1(\mathcal{A})$ is a *strong* universal principal connection if the following conditions hold:

$$\begin{aligned} \omega(1) &= 0, \\ \Delta_R \circ \omega &= (\omega \otimes \text{id}) \circ \text{Ad}_R \text{ (right H-colinearity)}, \\ (m \otimes \text{id}) \circ (\text{id} \otimes \Delta_R) \circ \omega &= 1 \otimes (\text{id} - \varepsilon) \text{ (fundamental vector condition)}, \\ d_u(a) - a_{(0)}\omega(a_{(1)}) &\in (\Omega_u^1(\mathcal{B})) \mathcal{A}, \quad \forall a \in \mathcal{A} \text{ (strongness)}. \end{aligned}$$

As in the case of spectral geometry it will be more convenient to use action of the $U(1)$ group rather than the coaction of the algebra of functions over $U(1)$. Since as the algebra of functions on $U(1)$ we consider the space of polynomials, and effectively we work with homogeneous elements $a \in \mathcal{A}^{(k)} \subset \mathcal{A}$ of a fixed degree k , which are defined as follows:

$$a \in \mathcal{A}^{(k)} \Leftrightarrow \Delta(a) = a \otimes z^k,$$

we can easily reformulate all conditions above using the language of $U(1)$ action, where we have:

$$a \in \mathcal{A}^{(k)} \Leftrightarrow e^{i\phi} \triangleright a = e^{ik\phi} a.$$

Definition 3.3. For a $U(1)$ Hopf-Galois extension $\mathcal{B} \hookrightarrow \mathcal{A}$ we say that the map $\omega : C^\infty(U(1)) \rightarrow \Omega_u^1(\mathcal{A})$ is a *strong* universal principal connection iff:

$$\begin{aligned} \omega(1) &= 0, \\ g \triangleright \omega &= \omega, \quad \forall g \in U(1), \\ m \circ (\text{id} \otimes \pi_n) \omega(z^k) &= \delta_{kn} - \delta_{n0}, \quad \forall k, n \in \mathbb{Z}, \\ d_u(a) - a\omega(z^k) &\in (\Omega^1(\mathcal{B})) \mathcal{A}, \quad \forall a \in \mathcal{A}^{(k)}. \end{aligned} \tag{3.2}$$

Here π_n projects an element on the part of a fixed homogeneity degree n .

It is possible to extend this definition of connections for nonuniversal differential calculi, however only after requiring certain compatibility conditions between the differential calculus on \mathcal{A} and a given calculus over the Hopf algebra H . Choosing a subbimodule $\mathcal{N} \subset \mathcal{A} \otimes \mathcal{A}$ we have an associated first order differential calculus over \mathcal{A} . If the canonical map χ maps \mathcal{N} to $\mathcal{A} \otimes Q$, where $Q \subset \ker \varepsilon \subset H$ is an Ad -invariant vector space then it is possible to use a calculus over H determined by Q using the Woronowicz construction of bicovariant calculi [16]. For details see [3, 12, 13].

In what follows we shall need the fact (see [14], p.251) that the in the case of Hopf-Galois extensions the multiplication map gives the natural isomorphism:

$$(\Omega^1(\mathcal{B})) \otimes_{\mathcal{B}} \mathcal{A} \approx (\Omega^1(\mathcal{B})) \mathcal{A}. \tag{3.3}$$

4 Spectral triples over noncommutative $U(1)$ bundles

We assume that there exists a real spectral triple over \mathcal{A} (for details on real spectral triples, notation and basic properties we refer to the textbook [11]), which is $U(1)$ equivariant, that is the action of $U(1)$ extends to the Hilbert space and the representation, the Dirac operator and the reality structure are $U(1)$ equivariant.

We denote by π the representation of \mathcal{A} on \mathcal{H} , D is the Dirac operator and J the reality structure.

Let δ be the operator on \mathcal{H} which generates the action of $U(1)$ on the Hilbert space. The $U(1)$ equivariance of the reality structure and D means that:

$$J\delta = -\delta J, \quad D\delta = \delta D, \quad (4.1)$$

whereas the equivariance of the representation is:

$$[\delta, \pi(a)] = \pi(\delta(a)), \quad \forall a \in \mathcal{A},$$

where $\delta(a)$ is the derivation of a arising from the $U(1)$ action.

For simplicity, we take the dimension of the spectral triple over \mathcal{A} to be odd, then the dimension of spectral triple over \mathcal{B} is even (in particular, the spectral triple over \mathcal{B} has a \mathbb{Z}_2 grading). We shall require that the signs ϵ, ϵ' present in $J^2 = \epsilon$ and $JD = \epsilon' DJ$, are not changed when we pass to the quotient. This corresponds, specifically, to the case of the top dimension 3 and the dimension of the quotient 2. Our example will be therefore three-dimensional, we have:

$$DJ = JD, \quad J^2 = -1.$$

Let us finally recall that J satisfies $[\pi(a), J\pi(b^*)J^{-1}] = 0$ and the order one condition

$$[[D, \pi(a)], J\pi(b^*)J^{-1}] = 0, \quad \forall a, b \in \mathcal{A}, \quad (4.2)$$

which will be important in further considerations.

Throughout the rest of the paper we omit writing π , whenever it is clear from the context that we mean the image of $a \in \mathcal{A}$ by the chosen representation π .

4.1 Projectable spectral triples

We start by assuming the existence of an additional structure on the spectral triple.

Definition 4.1. We say that the $U(1)$ equivariant spectral triple $(\mathcal{A}, D, J, \mathcal{H}, \delta)$ is *projectable* along the fibres if there exists an operator Γ , a \mathbb{Z}_2 grading of the Hilbert space \mathcal{H} , which satisfies the following conditions:

$$\begin{aligned} \forall a \in \mathcal{A} : [\Gamma, a] &= 0, \\ \Gamma J &= -J\Gamma, \quad \Gamma \delta = \delta \Gamma, \quad \Gamma^* = \Gamma, \quad \Gamma^2 = \text{id}, \end{aligned} \quad (4.3)$$

and define the *horizontal Dirac operator* as

$$D_h = \frac{1}{2} \Gamma [\Gamma, D].$$

Remark 1. Note that the operator D_h anticommutes with Γ . It will be employed for construction of even spectral triples over \mathcal{B} . Note also that the signs in the definition are adjusted to the case of dimension 3 bundle over a 2-dimensional space.

Although classically this follows, we shall impose now certain requirements in order to guarantee that the differential calculus over \mathcal{B} does not depend on the choice of projection.

Definition 4.2. We say that the differential calculus $\Omega_D^1(\mathcal{B})$ on \mathcal{A} is *projectable* along the fibres if

$$[D_h, b] = [D, b], \quad \forall b \in \mathcal{B}. \quad (4.4)$$

This property tells us that the operators representing the one-forms over \mathcal{B} generated by D_h and by D are the same. Moreover, it follows then that the restriction of $[D, b]$ to \mathcal{H}_0 is just equal to the operator $[D_0, b]$ on \mathcal{H}_0 .

Next, we shall make one more additional assumption, which in the classical case amounts to the situation when the $U(1)$ fibres are of equal length. What we propose, is a geometric characterization of the Dirac operator, which closely follows the analysis of Amman and Bär [2].

Definition 4.3. We say that the $U(1)$ bundle has fibres of constant length (taken to be $2\pi\ell$) if D_v (called the vertical part of the Dirac operator),

$$D_v = \frac{1}{\ell} \Gamma \delta,$$

is such that:

$$Z = D - D_h - D_v,$$

is a bounded operator such that Z commutes with the elements from the commutant:

$$[Z, Ja^*J^{-1}] = 0, \quad \forall a \in \mathcal{A}. \quad (4.5)$$

Remark 2. Note that Z has to commute with Γ and also with \mathcal{B} , due to the condition (4.5) and (4.4). We observe also that the requirement of the boundedness of Z can be reinforced by requiring that Z is operator of zero order in the sense of generalized pseudodifferential operators of Connes and Moscovici [7].

We define the space $\mathcal{H}_k \subset \mathcal{H}$, $k \in \mathbb{Z}$, to be a subspace of vectors homogeneous of degree k in \mathcal{H} that is, they are eigenvectors of δ of eigenvalue k . The relation (4.1) means that

$$J\mathcal{H}_k = \mathcal{H}_{-k}.$$

In particular the subspace \mathcal{H}_0 is J invariant. From the equivariance of D we see that each \mathcal{H}_k is preserved by the action of D :

$$D\mathcal{H}_k \subset \mathcal{H}_k.$$

Since $[\delta, D_h] = 0$ we see that D_h preserves each of the subspaces \mathcal{H}_k . We shall denote by D_k its restriction to each subspace \mathcal{H}_k . Similarly, we denote by γ_k the restriction of Γ to \mathcal{H}_k and by j_k the restriction of J (j_k is a map $\mathcal{H}_k \rightarrow \mathcal{H}_{-k}$). Now we have:

Proposition 4.4. *The operators D_k, γ_k, j_k satisfy the commutation relations*

$$\gamma_k D_k = -D_k \gamma_k, \quad j_k D_k = D_{-k} j_k, \quad j_k \gamma_k = -\gamma_{-k} j_k. \quad (4.6)$$

The data $(\mathcal{B}, \mathcal{H}_0, D_0, \gamma_0, j_0)$ give an even real spectral triple of KR-dimension 2 over \mathcal{B} . For $k \neq 0$, $(\mathcal{B}, \mathcal{H}_k, D_k, \gamma_k)$ are even spectral triples, which are pairwise real.

Proof. Clearly D_h is a selfadjoint operator, which has the same commutation relation with J and Γ as D ¹. Therefore, the relations (4.6) follow after restriction to subspaces.

That each D_k has bounded commutator with the elements from \mathcal{B} is an immediate consequence of the fact that it is a restriction of D_h , which has this property. To see that D_k has a compact resolvent observe that $D - Z$, which is a bounded perturbation of D , is again $U(1)$ invariant. and we can restrict it to \mathcal{H}_k . Its eigenvalues are:

$$\pm \sqrt{k^2/\ell^2 + \lambda_{(k)}^2}$$

where $\lambda_{(k)}$ are eigenvalues of D_k . Since D has compact resolvent, $|D_k|$ must have pure spectrum diverging to ∞ and so D_k also has a compact resolvent.

The KR-dimension follows from the relations (4.6) and $J^2 = -1$. \square

Now, few remarks are in order.

Remark 3. Note that taking a pair $\mathcal{H}_k \oplus \mathcal{H}_{-k}$ yields again a real even spectral triple, which is, however, reducible (in the sense of [15]). We shall see in the next section that each triple built on a single \mathcal{H}_k is in fact a spectral triple twisted by a module over \mathcal{B} .

¹In the case of dimension other than 3 it is possible to adjust the signs in the definition of Γ and j_k so that the resulting KR-dimension of the projected spectral triple shall be correct.

Remark 4. As follows from the proof of Proposition 4.4 the asymptotic spectral properties of D_k are the same as properties of D restricted to \mathcal{H}_k . Therefore spectral dimension of each D_k can be at most the same as that of D , which however does not imply that it is exactly 2 as we know in the classical case.

Remark 5. In the classical situation, when one is able to consider the fibres over points of the base space, there is no problem to define the length of a fibre and, consequently, to restrict the considerations to the case when all fibres are of equal length. In the general noncommutative setup, this is no longer possible. Instead, we have proposed in the definition 4.3 above how to replace and extend this property in a way which links the length of fibres to the form of the Dirac operator. There may be, however, some other alternatives. We mention here just one other possible definition, which will work in the case of finitely summable spectral triples of metric dimension n .

We can say that the $U(1)$ bundle \mathcal{A} with the equivariant spectral triple and the Dirac D has fibres of length ℓ , if the restriction of D to a $U(1)$ invariant subspace of \mathcal{H} is an operator of spectral dimension $(n - 1)$ and for any element $b \in \mathcal{B}$:

$$\oint b|D|^{-n} = \ell \oint b|D_0|^{-n+1},$$

where D_0 is the restriction of D to the invariant Hilbert subspace \mathcal{H}_0 . The symbol \oint denotes the noncommutative integral:

$$\oint T := \operatorname{Res}_{s=0} \zeta_D^T(s) = \operatorname{Res}_{s=0} \operatorname{Tr} (T|D|^{-s}).$$

Although in the classical case this definition implies that the fibres are indeed of equal length, in the noncommutative case it is far from being clear. The advantage of this definition, is that it is not sensitive to the bounded perturbations of the Dirac operator, which do not commute with the algebra elements.

From now on, for simplicity we assume $\ell = 1$.

4.2 Spectral triples twisted by a module

In this subsection we shall discuss how to twist real spectral triples by a left-module equipped with a hermitian connection.

Instead of using the usual algebraic definition of connection, we shall use the operator language suited best for the further considerations. Let $(\mathcal{B}, \mathcal{H}, D, J)$ be a real spectral triple over an algebra \mathcal{B} . Let \mathcal{H}_M be another Hilbert space with a representation of \mathcal{B} . Let M be space of (some) \mathcal{B} -linear maps $\mathcal{H} \rightarrow \mathcal{H}_M$ such that

(i): $M(\mathcal{H}) = \mathcal{H}M$ is dense in \mathcal{H}_M and

(ii): the multiplication map from $\mathcal{H} \otimes_{\mathcal{B}} M$ to $\mathcal{H}M$ is an isomorphism.

Using the right \mathcal{B} -module structure on \mathcal{H} , $hb = Jb^*J^{-1}h$, $\forall b \in \mathcal{B}, h \in \mathcal{H}$, we introduce a left \mathcal{B} -module structure on M through:

$$(bm)(h) = m(hb), \quad \forall m \in M.$$

We find it convenient to write the action of m on h from the right, i.e. hm instead of $m(h)$, then the left \mathcal{B} -linearity of $m \in M$ reads

$$(bh)m = b(hm),$$

while the left \mathcal{B} action on M becomes

$$h(bm) = (hb)m, \quad \forall h \in \mathcal{H}, b \in \mathcal{B}.$$

It follows from the order one condition that there is a right action of $\Omega_D^1(\mathcal{B})$ on \mathcal{H} ,

$$h\omega := -J\omega^*J^{-1}h, \quad (4.7)$$

which is left \mathcal{B} -linear, where ω^* is the adjoint of the operator ω so that:

$$([D, b])^* = -[D, b^*],$$

and

$$h([D, b]) = D(hb) - (Dh)b, \quad \forall h \in \mathcal{H}, b \in \mathcal{B}.$$

This induces a left action of $\Omega_D^1(\mathcal{B})$ on M , and $\Omega_D^1(\mathcal{B}) \bullet M$ is just the space of all compositions $m \circ \omega$ of left \mathcal{B} -linear maps: (right action of) $\omega \in \Omega_D^1(\mathcal{B})$ and $m \in M$. Note that the action of such a composition on h , when written from the right, becomes

$$(m \circ \omega)(h) = h(\omega m) = (h\omega)m.$$

It is easy to verify that the composition is compatible with the right \mathcal{B} action on $\Omega_D^1(\mathcal{B})$ and the left \mathcal{B} action on M

$$h(\omega b)m = h\omega(bm), \quad \forall b \in \mathcal{B}, m \in M, h \in \mathcal{H}.$$

Next we pass to connections (covariant derivatives) on M . Usually they would take values in $\Omega_D^1(\mathcal{B}) \otimes M$, but inspired by (3.3) with its right side regarded as operators we propose the following definition.

Definition 4.5. We call a linear map $\nabla : M \rightarrow \Omega_D^1(\mathcal{B}) \bullet M$ a *D-connection* on M , if it satisfies the Leibniz rule

$$\nabla(bm) = [D, b]m + b\nabla(m), \quad \forall b \in \mathcal{B}, m \in M.$$

We say that the connection is *hermitian* if for each $m_1, m_2 \in M$

- as an operator on \mathcal{H} , $m_1^\dagger \circ m_2 \in J\mathcal{B}J^{-1}$, (so it is in the commutant of \mathcal{B});
- writing the actions on arbitrary $h \in \mathcal{H}$ from the right:

$$h\nabla(m_2)m_1^\dagger - hm_2\nabla(m_1)^\dagger = (Dh)m_2m_1^\dagger - D(hm_2m_1^\dagger). \quad (4.8)$$

Note that D -connection (which could be also called a covariant D -derivative) over the module M uses the differential calculus, which comes from the spectral triple. Note also that due to the first condition M cannot consist of all possible \mathcal{B} -linear maps, and in particular they have to preserve the domain of D .

Finally, using a hermitian D -connection and the properties of M we define an (unbounded) twisted Dirac operator D_M .

Definition 4.6. Define an operator D_M over the the dense domain in \mathcal{H}_M by

$$D_M(hm) = (Dh)m + h\nabla(m), \quad \forall h \in \mathcal{H}, m \in M. \quad (4.9)$$

The denseness follows from the assumption i) on M , while to see that D_M is well defined, i.e. it depends only on the product hm , due to the assumption ii) it suffices to check that for $b \in \mathcal{B}$

$$(D(hb))m + hb\nabla(m) = (Dh)bm + h\nabla(bm),$$

which follows from the Leibniz rule for ∇ .

Proposition 4.7. *The operator D_M is selfadjoint, has bounded commutators with \mathcal{B} and compact resolvent.*

Proof. We show that D_M is symmetric

$$\begin{aligned} (h_1m_1, D_M(h_2m_2)) &= (h_1m_1, (Dh_2)m_2) + (h_1m_1, h_2\nabla(m_2)) \\ &= \left(h_1, (Dh_2)m_2m_1^\dagger\right) + \left(h_1, h_2\nabla(m_2)m_1^\dagger\right) \\ &= \left(h_1, D(h_2m_2m_1^\dagger)\right) + \left(h_1, h_2m_2\nabla(m_1)^\dagger\right) \\ &= ((Dh_1)m_1, h_2m_2) + (h_1\nabla(m_1), h_2m_2) \\ &= (D_M(h_1m_1), h_2m_2), \end{aligned}$$

where in the third equality we used (4.8). However, as $m \in M$ and $\nabla(m)$ are all bounded operators and D is selfadjoint then it is clear that D_M is selfadjoint as well.

Next we compute the commutator with $b \in \mathcal{B}$,

$$\begin{aligned} [D_M, b](hm) &= D_M(bhm) - bD_M(hm) \\ &= (Dbh)m + (bh)\nabla(m) - b(Dh)m - b(h\nabla(m)) = ([D, b]h)m, \end{aligned}$$

and hence $||[D_M, b]|| \leq ||[D, b]||$.

We pass to show that D_M has compact resolvent. When M is a finite free module over \mathcal{B} with the basis m_i over B we have (summation over j implied)

$$D_M(h_j m_j) = (Dh_j)m_j + h_j \nabla(m_j).$$

Now, $\nabla(m_j)$ can be written as $\omega_{j,k} m_k$ and hence the second part of the expression is in fact a bounded operator on \mathcal{H}_M . Therefore, the spectrum of D_M is at most a spectrum of a bounded perturbation of D , which has a compact resolvent, so must have D_M . Similar discussion applies to the case when M is a finite projective module over \mathcal{B} . \square

5 Connections

In order to define a strong $U(1)$ connection over the bundle using the differential calculus given by the Dirac operator, we need to impose certain conditions on the Dirac operator itself.

Definition 5.1. We say that the first order differential calculus $\Omega_D^1(\mathcal{A})$, over \mathcal{A} given by the Dirac operator D is *compatible* with the standard de Rham calculus over $U(1)$ if the following holds:

$$\forall p_i, q_i \in \mathcal{A} : \sum_i p_i [D, q_i] = 0 \Rightarrow \sum_i p_i \delta(q_i) = 0. \quad (5.1)$$

Recalling the standard characterization of the first order de Rham calculus over an algebra of functions on a group via $Q = (\ker \varepsilon)^2$, the compatibility as discussed at the end of Sect.3 follows from the next lemma.

Let the first order differential calculus over \mathcal{A} be given by a subbimodule \mathcal{N} :

$$\mathcal{N} \subset \ker m \subset \mathcal{A} \otimes \mathcal{A},$$

defined by the relation

$$\sum_i p_i \otimes q_i \in \mathcal{N} \Leftrightarrow \sum_i p_i [D, q_i] = 0.$$

Lemma 5.2. *The image of \mathcal{N} by the canonical map χ is in $\mathcal{A} \otimes (\ker \varepsilon)^2$.*

Proof. Let $\sum_i p_i \otimes q_i \in \mathcal{N}$. Decompose q_i as a sum of homogeneous elements of δ :

$$q_i = \sum_n q_i^{(n)}.$$

Then since $\sum_i p_i \otimes q_i \in \ker m$, using the identity (5.1) we have,

$$\sum_{i,n} p_i q_i^{(n)} = 0, \quad \sum_{i,n} n p_i q_i^{(n)} = 0.$$

which we can solve for $p_i q_i^{(0)}$ and $p_i q_i^{(1)}$:

$$\begin{aligned} \sum_i p_i q_i^{(1)} &= - \sum_{i,n \neq 1} n p_i q_i^{(n)}, \\ \sum_i p_i q_i^{(0)} &= - \sum_{i,n \neq 0} p_i q_i^{(n)} = \sum_{i,n \neq 0,1} (n-1) p_i q_i^{(n)}. \end{aligned}$$

Applying canonical map χ to $\sum p_i \otimes q_i$ we obtain:

$$\begin{aligned} \chi\left(\sum_i p_i \otimes q_i\right) &= \sum_{i,n} p_i q_i^{(n)} \otimes z^n \\ &= \sum_{i,n \neq 0,1} p_i q_i^{(n)} \otimes (z^n - 1 - n(z-1)). \end{aligned}$$

The second factor on the right-hand side is in $(\ker \varepsilon)^2$ since it can be written as

$$(z-1)(z^{n-1} + \cdots + z + 1 - n).$$

□

Now, we are ready to define a strong connection for a principal $U(1)$ bundle with a differential calculus set by the Dirac operator.

Definition 5.3. Let $\Omega_D^1(\mathcal{A})$ be as in definition 5.1. We say that $\omega \in \Omega_D^1(\mathcal{A})$ is a strong principal connection for the $U(1)$ bundle $\mathcal{B} \hookrightarrow \mathcal{A}$ if the following conditions hold:

$$[\delta, \omega] = 0, \quad (U(1) \text{ invariance of } \omega)$$

$$\text{if } \omega = \sum_i p_i [D, q_i] \text{ then } \sum_i p_i \delta(q_i) = 1, \quad (\text{vertical field condition}),$$

$$\forall a \in \mathcal{A} : [D, a] - \delta(a)\omega \in \Omega_D^1(\mathcal{B})\mathcal{A}, \quad (\text{strongness})$$

Observe that the second condition (which in the classical case corresponds to the value of ω on fundamental vertical vector field) makes sense due to assumption (5.1). Our definition is motivated by the classical example of adapting the definition 3.3 to the case of the de Rham calculus over a $U(1)$ bundle. In such a situation one has $\omega(z^n) = n\omega(z)$ for the functions on $U(1)$, therefore the strong principal connection is completely defined by $\omega = \omega(z)$.

We shall see in section 5 that the fourth condition (strongness) will play a significant role in the extension of Dirac operator.

5.1 Lifting the Dirac operator through connection

In this section we shall now put together the construction from the section 4.2 and the notion of strong connections from last section to obtain the construction of principal connections compatible with spectral triples. The role of the Hilbert space \mathcal{H} , the Hilbert space \mathcal{H}_M and the (sub)space M of operators from \mathcal{H} to \mathcal{H}_M (all of them left \mathcal{B} -modules) will be played by, respectively, the Hilbert space \mathcal{H}_0 , the Hilbert space \mathcal{H}_k and $\mathcal{A}^{(k)}$ (the eigenspaces of δ).

Here $\mathcal{A}^{(k)}$ is assumed to satisfy the properties (i) and (ii) required for M , and it acts on \mathcal{H}_0 from the right, or more precisely $M = J\mathcal{A}^{(k)*}J^{-1}$.

Proposition 5.4. *Let $(\mathcal{A}, D, \mathcal{H}, J)$ be the projectable real spectral triple of a $U(1)$ bundle with a projectable differential calculus $\Omega_D^1(\mathcal{B})$ as defined in section 4, and let ω be the strong connection in the sense of definition (5.3). Then the map:*

$$\nabla_\omega : \mathcal{A}^{(k)} \ni a \mapsto \nabla_\omega(a) := [D, a] - ka\omega,$$

where both a in the domain of ∇_ω , and its image $\nabla_\omega(a)$ are regarded as operators on \mathcal{H}_0 acting from the right, defines a D_0 -connection (as defined in (4.5)) over left \mathcal{B} -module $\mathcal{A}^{(k)}$.

Proof. First, observe that (thanks to the real structure) the elements of $\mathcal{A}^{(k)}$ indeed map \mathcal{H}_0 to \mathcal{H}_k as \mathcal{B} -linear morphism.

To see that ∇_ω is well-defined observe first that due to the strongness condition and the fact that ω is $U(1)$ -invariant the element $\nabla_\omega(a)$ is indeed in the restriction to \mathcal{H}_0 of $\Omega_D^1(\mathcal{B}) \bullet \mathcal{A}^{(k)}$. But due to the assumption that $[D, b] = [D_h, b]$ for $b \in \mathcal{B}$, the restriction of $\Omega_D^1(\mathcal{B})$ to \mathcal{H}_0 is just $\Omega_{D_0}^1(\mathcal{B})$. Hence we see that $\nabla_\omega(a)$ (regarded as an operator acting on \mathcal{H}_0 from the right) indeed belongs to $\Omega_D^1(\mathcal{B}) \bullet \mathcal{A}^{(k)}$ (in spite we used the commutator with D in its definition). Moreover ∇_ω satisfies the Leibniz rule:

$$\nabla_\omega(ba) = [D, ba] - kba\omega = [D_0, b]a + b\nabla_\omega(a).$$

This shows that ∇_ω is a D_0 -connection. □

Further, we have:

Lemma 5.5. *The D_0 -connection ∇_ω is hermitian if ω is selfadjoint (as an operator on \mathcal{H}).*

Proof. We compute (4.8) for $a_1, a_2 \in \mathcal{A}^{(k)}$, $h \in \mathcal{H}_0$, using the rule (4.7):

$$\begin{aligned} h \left(\nabla(a_2)a_1^\dagger - a_2\nabla(a_1)^\dagger - (Dh)a_2a_1^\dagger + D(ha_2a_1^\dagger) \right) \\ = h \left(([D, a_2] - ka_2\omega) a_1^\dagger - h \left(a_2([D, a_1] - ka_1\omega)^\dagger \right) - h[D, a_2a_1^\dagger] \right) \\ = h \left(ka_2(\omega^\dagger - \omega)a_1^\dagger \right). \end{aligned}$$

□

Definition 5.6. Let ω be a principal connection and ∇_ω the related D_0 -connection on $\mathcal{A}^{(k)}$, $k \in \mathbb{Z}$. We construct as in section 4.2 the twisted spectral triple $(\mathcal{B}, \mathcal{H}_k, D_\omega^{(k)})$, where the twisted Dirac operator $D_\omega^{(k)}$ on \mathcal{H}_k is the closure of $(D_0)_{\mathcal{A}^{(k)}}$ (see (4.9)) defined on a dense subset $\text{Dom}(D)\mathcal{A}^{(k)}$ of \mathcal{H}_k . Taking D_ω to be the closure of the direct sum of the operators $D_\omega^{(k)}$ for all $k \in \mathbb{Z}$, we obtain the twisted Dirac operator D_ω on \mathcal{H} .

Note that notwithstanding Proposition 4.7 valid for $D_\omega^{(k)}$, from the construction it is not entirely obvious that the Dirac operator D_ω is selfadjoint or has bounded commutators with the elements from the algebra \mathcal{A} . Therefore we have:

Proposition 5.7. *The twisted Dirac operator D_ω is selfadjoint if ω is a selfadjoint one form and has bounded commutators with all elements of \mathcal{A} .*

Proof. We compute the action of D_ω on an element hp in its domain, with $h \in \mathcal{H}_0$ and $p \in \mathcal{A}^{(n)}$:

$$\begin{aligned} D_\omega(hp) &= (D_0h)p + h[D, p] - nhp\omega \\ &= (D_0h)p + [D, Jp^*J^{-1}]h + J\omega^*J^{-1}nhp \\ &= D(hp) + ((D_0 - D)h)p + J\omega^*J^{-1}\delta(hp) \\ &= (D + J\omega^*J^{-1}\delta - Z)(hp), \end{aligned}$$

where we have used the decomposition of D into the horizontal part (which restricted to \mathcal{H}_0 is D_0), the vertical part (which vanishes on \mathcal{H}_0) and the bounded perturbation Z , which by (4.5) obeys $(Zh)p = Z(hp)$.

Hence we see that D_ω on \mathcal{H}_n is a reduction of the operator $D + J\omega^*J^{-1}\delta + Z$ to this subspace, so the latter equals the direct sum of the reductions. Now, D and δ

are selfadjoint by definition, Z and ω are by assumption bounded and selfadjoint. Since δ is relatively bounded with respect to D , by Kato-Rellich theorem D_ω is selfadjoint on \mathcal{H} .

To see that the commutators with the elements from \mathcal{A} are bounded we observe that D has bounded commutators with each $a \in \mathcal{A}$ and since ω is a one-form, from the order one condition the commutator of the second term with a is $J\omega^*J^{-1}\delta(a)$ and hence it is bounded. The third term, as a commutator of two bounded elements remains bounded. Hence, $[D_\omega, a]$ is bounded for any a . \square

Observe that strongness is necessary to define a D_0 connection and then, in turn, to twist the Dirac operator by ω on every module $\mathcal{A}^{(k)}$. On the other hand, the vertical field condition assures that the D_ω is indeed a horizontal part of some Dirac operator on \mathcal{A} . However, for that we shall need a stronger condition on the bounded operator Z than its commutativity with \mathcal{B} (and with Γ), namely that it commutes with \mathcal{A} .

Proposition 5.8. *Let Z be as in sect. 4.1. Let*

$$\mathcal{D}_\omega = \Gamma\delta + D_\omega. \quad (5.2)$$

If Z commutes with the algebra \mathcal{A} then $(\mathcal{A}, \mathcal{D}_\omega, \mathcal{H})$ is a projectable spectral triple with equal length fibres and the horizontal part of the operator \mathcal{D}_ω coincides with D_ω .

Proof. The operator \mathcal{D}_ω can be written as:

$$\mathcal{D}_\omega = D + (J\omega J^{-1} + \Gamma)\delta + Z.$$

Then using similar arguments as in the case of D_ω we see that it is selfadjoint and has bounded commutators with the elements of \mathcal{A} .

We show now that \mathcal{D}_ω has a compact resolvent. Since D_ω is a twist of D_0 by a connection, then by the arguments from section 4.2 it has a compact resolvent when restricted to each $\mathcal{H}^{(n)}$. Since D_ω then anticommutes with Γ , squaring \mathcal{D}_ω we obtain:

$$\mathcal{D}_\omega^2 = D_\omega^2 + \delta^2.$$

Now, it is easy to check that the spectrum of this operator consists of the values of the form $n^2 + \lambda_n^2$, where λ_n is in the spectrum of D_ω , restricted to $\mathcal{H}^{(n)}$. Since for each n the sequence $\{\lambda_n\}$ cannot have an accumulation point other than infinity, we see that it must be true also for $\{n^2 + \lambda_n^2\}$.

This shows that $(\mathcal{A}, \mathcal{D}_\omega, \mathcal{H})$ is a spectral triple (not necessarily real). It is clearly projectable with the original grading Γ .

We compute now the horizontal part of \mathcal{D}_ω :

$$\begin{aligned}
(\mathcal{D}_\omega)_h h &= \frac{1}{2} \Gamma [\Gamma, \mathcal{D}_\omega] h = \frac{1}{2} \Gamma [\Gamma, D_\omega] h \\
&= \frac{1}{2} \Gamma [\Gamma, (D + J\omega J^{-1} \delta - Z)] h \\
&= \left(D_h + J \left(\frac{1}{2} \Gamma [\Gamma, \omega] \right) J^{-1} \delta \right) h \\
&= (D_h + J\omega J^{-1} \delta + \Gamma \delta) h \\
&= D_\omega h.
\end{aligned} \tag{5.3}$$

Here we used that for a general $\omega = \sum a_i [D, a'_i]$, we have:

$$\omega = \sum a_i ([D_h, a'_i] + [\Gamma \delta, a'_i] + [Z, a'_i]).$$

The first term anticommutes with Γ whereas the other two commute by construction. If, as assumed Z commutes with the elements of the algebra then

$$\omega = \sum a_i [D_h, a'_i] + \sum \Gamma a_i \delta(a'_i)$$

and using the vertical field condition we have:

$$\omega = \frac{1}{2} \Gamma [\Gamma, \omega] + \Gamma.$$

□

Observe that if we do not assume that Z commutes with \mathcal{A} (note that this together with (4.5) is a severe limitation for Z) we cannot have the property $(\mathcal{D}_\omega)_h = D_h$. Nevertheless, still the spectral triple will be projectable, the restriction of both operators to \mathcal{H}_0 and the resulting spectral triple over \mathcal{B} will be identical. However, without that assumption we cannot obtain a spectral triple with fibres of equal length.

We recapitulate our constructions so far as the following sequence of steps: start with a full Dirac operator of a projectable real spectral triple, take a strong connection, define a D -connection and then twist by it the projected Dirac operator, and finally add to it $\Gamma \delta$. This produces a new Dirac-type operator \mathcal{D}_ω over \mathcal{A} . Of course, it is just one of the class of other possible Dirac operators, since we may take any bounded perturbation of \mathcal{D}_ω . For the setup we have just discussed it is worth to formulate the following definition:

Definition 5.9. We say that the connection ω is *compatible* with the Dirac operator D if D_ω and D_h coincide on a dense subset of \mathcal{H} .

Following results of [2] it is not difficult to see that the classical Dirac operator, which arises from the metric compatible with a $U(1)$ principal connection (in the sense of the construction in the section 2) is indeed compatible with this connection in the sense of the above definition, whereas the operator \mathcal{D}_ω differs from it by a torsion term.

It is worth to mention that in the genuine noncommutative situation we don't expect Z to commute with \mathcal{A} , in this case \mathcal{D}_ω given by (5.2) and D_ω would still coincide up to some bounded perturbation.

6 The noncommutative torus

To see how the definitions work in a noncommutative case, we study in detail the case of the 3-dimensional noncommutative torus as a $U(1)$ bundle over the 2-dimensional noncommutative torus.

We choose the generators of the \mathbb{T}_θ^3 to be $U_i, i = 1, 2, 3$, with the relations

$$U_j U_k = e^{2\pi i \theta_{jk}} U_k U_j,$$

where θ_{jk} is an antisymmetric real matrix. We assume that neither of its entries is rational. An element of the algebra of smooth functions on the noncommutative torus is of the form:

$$a = \sum_{k,l,m \in \mathbb{Z}} \alpha_{klm} U_1^k U_2^l U_3^m,$$

where $\alpha_{k,l,m}$ is a rapidly decreasing sequence. The canonical trace on the algebra is:

$$\tau(a) = \alpha_{000}.$$

We start with the canonical Hilbert space \mathcal{H}_0 of the GNS construction with respect to the trace τ on \mathbb{T}_θ^3 , and π the associated faithful representation. With $e_{k,l,m}$ the orthonormal basis of \mathcal{H}_0 we have:

$$\begin{aligned} U_1 e_{k,l,m} &= e_{(k+1),l,m}, \\ U_2 e_{k,l,m} &= e^{2\pi i k \theta_{21}} e_{k,(l+1),m}, \\ U_3 e_{k,l,m} &= e^{2\pi i (k \theta_{31} + l \theta_{32})} e_{k,l,(m+1)}, \end{aligned}$$

where k, l, m are in \mathbb{Z} or $\mathbb{Z} + \frac{1}{2}$ depending on the choice of the spin structure. The projectable spin structures must have the trivial spin structure on the fibre, so $m \in \mathbb{Z}$, which we assume from now on.

We double the Hilbert space taking $\mathcal{H} = \mathcal{H}_0 \otimes \mathbb{C}^2$, with the diagonal representation of the algebra. The canonical equivariant spectral triple over \mathbb{T}_θ^3 is given by the Dirac operator D and the reality structure J of the form:

$$D = \sum_{j=1}^3 \sigma^j \delta_j, \quad J = i\sigma^2 \circ J_0, \quad (6.1)$$

where σ^i are selfadjoint Pauli matrices [11]. J_0 here is the canonical Tomita-Takesaki antilinear map on the Hilbert space \mathcal{H}_0 .

$$J_0 e_{k,l,m} = e_{-k,-l,-m},$$

and δ_i are selfadjoint derivations acting diagonally on the basis:

$$\delta_1 e_{k,l,m} = k e_{k,l,m}, \quad \delta_2 e_{k,l,m} = l e_{k,l,m}, \quad \delta_3 e_{k,l,m} = m e_{k,l,m}.$$

Observe that $JD = DJ$ as required in the spectral triple of dimension 3. We choose the following $U(1)$ action on the torus \mathbb{T}_θ^3 , defined through the action on the generators:

$$e^{i\phi} \triangleright U_1 = U_1, \quad e^{i\phi} \triangleright U_2 = U_2, \quad e^{i\phi} \triangleright U_3 = e^{i\phi} U_3, \quad (6.2)$$

and the induced diagonal action on the Hilbert space:

$$e^{i\phi} \triangleright e_{k,l,m} = e^{im\phi} e_{k,l,m},$$

The $U(1)$ invariant subalgebra is generated by U_1 and U_2 , and can be identified with the 2-dimensional noncommutative torus \mathbb{T}_θ^2 , where the indices of θ_{ij} run over 1, 2. It is straightforward to check that \mathbb{T}_θ^3 is a Hopf-Galois extension of \mathbb{T}_θ^2 (when θ_{12} match).

The chosen Dirac operator is one which is fully equivariant, that is invariant under all three $U(1)$ symmetries. This is not necessary in our case, as we need only one $U(1)$ invariance. In particular, we shall allow for the coupling of D to gauge fields (or for the 'internal fluctuations' in terminology of [6]).

Remark 6. The space of possible fluctuations of the Dirac operator of the gauge connection type (by one-forms) is given by:

$$D_A = D + \sigma^i A_i + J(\sigma^i A_i)J^{-1},$$

where $A_i \in \mathbb{T}_\theta^3$, $i = 1, 2, 3$, satisfy: $A_i = A_i^*$. Since D_A is a bounded perturbation of D it is not difficult to see that $\mathcal{A}, \mathcal{H}, D_A$, is a (real) spectral triple.

In the sequel we further require that the Dirac operator D_A is $U(1)$ -invariant and thus we restrict A_i to belong to the invariant subalgebra \mathbb{T}_θ^2 .

Before we proceed, observe that any one-form in $\Omega_{D_A}^1$ is (trivially) a one-form in Ω_D^1 . This provides us with a more convenient description of the bimodule of one-forms. We begin with:

Lemma 6.1. *The differential calculus generated by D_A satisfies the compatibility condition from the definition 5.1 if $A_3 = 0$.*

Proof. Take p_i, q_i such that $\sum_i p_i [D, q_i] = 0$. Since σ^i are linearly independent, we have for σ^3 :

$$\sum_i p_i (\delta_3(q_i) + [A_3, q_i]).$$

If $A_3 = 0$ then the condition follows. \square

This condition on A agrees with the next Lemma on projectability.

Lemma 6.2. *If $A_3 = 0$ then there exists a unique operator*

$$\Gamma = \sigma^3,$$

which makes the spectral geometry over \mathbb{T}_θ^3 projectable (in the sense of definition 4.1) with constant length fibres.

Proof. It is easy to see that indeed $\Gamma = \sigma^3$ does satisfy the projectability and equal length fibers. Conversely, any operator Γ , which commutes with the algebra, is $U(1)$ -invariant, anticommutes with J and is a \mathbb{Z}_2 grading must be a linear combination of Pauli matrices. Then, the requirement 4.1 that D has equal length fibres fixes Γ to be σ^3 . \square

Furthermore with we have other essential properties discussed in sect. 4.1.

Lemma 6.3. *If $A_3 = 0$ and $\Gamma = \sigma^3$, then the horizontal part of D_A restricted to \mathcal{H}_k gives an even spectral triple over the two-dimensional torus \mathbb{T}_θ^2 , which is real if $k = 0$ and pairwise real if $\pm k \in \mathbb{N}$, and the differential calculus $\Omega_{D_A}(\mathcal{A})$ is projectable in the sense of Definition 4.2.*

Proof. It is not difficult to recognize that $(D_A)_h$ restricted to \mathcal{H}_k is a fluctuated Dirac operator over the two-torus. To see that for every $b \in \mathbb{T}_\theta^2$ we have on a dense domain in \mathcal{H} :

$$[D_A, b] = [(D_A)_h, b],$$

it is sufficient to see that $\delta(b) = 0$ and since $A_3 = 0$, only the horizontal part has non-trivial commutators. Furthermore, since both A_1 and A_2 must be $U(1)$ invariant, they are, in fact elements of the invariant algebra \mathbb{T}_θ^2 . Therefore, a commutator

of $(D_A)_h$ (or D_A) with $b \in \mathbb{T}_\theta^2$ is in fact a one-form, which contains only elements from \mathbb{T}_θ^2 . Hence, the restriction to \mathcal{H}_0 is an isomorphism and therefore the bimodule of one-forms over \mathbb{T}_θ^2 generated by $(D_A)_0$ is the same as the one generated by $(D_A)_h$. \square

Note that, so far, there is a remarkable consistency in all the conditions that we are imposing on the Dirac operator and the spectral geometry of the noncommutative three-torus, viewed as a $U(1)$ bundle over the noncommutative two-torus. But we find that D_A with $A_3 = 0$ satisfies also the condition of equal length fibres proposed in the remark 5.

Lemma 6.4. *The Dirac operator D_A , with $A_3 = 0$ satisfies*

$$\oint b|D_A|^{-3} = \oint b|(D_A)_0|^{-2}.$$

Proof. Indeed, using the results of the explicit calculations of the spectral action [9], it can be seen that the noncommutative integral of the perturbed Dirac operator $|D_A|^{-3}$ does not depend on A . Hence, we can work with D alone and its restriction to \mathcal{H}_0 . Then, the proof is reduced to a simple exercise. \square

We mention that an easy check suggests that we could have taken remark 5 as a defining condition for Γ (at least in that case).

As a matter of fact we have moreover the following analogue of yet another classical property: orthogonality condition of the base and the fibers:

Lemma 6.5. *The Dirac operator D_A with $A_3 = 0$ and $\Gamma = \sigma^3$ satisfy*

$$\oint \Gamma \rho |D_A|^{-3} = 0, \quad \forall \rho \in \Omega_D^1(\mathbb{T}_\theta^2). \quad (6.3)$$

In fact, $\Gamma = \sigma^3$ is a unique (up to sign) $U(1)$ -invariant \mathbb{Z}_2 -grading of the Hilbert space, which commutes with the algebra and its commutant and satisfies (6.3).

Proof. Take an arbitrary one form $\Gamma = \sum \sigma^i \rho_i$. From the orthogonality requirement (6.3), taking as $\eta = b\sigma^1$ and $b\sigma^2$ respectively, for $b \in \mathbb{T}_\theta^2$ we obtain:

$$\oint \rho_1 b |D|^{-3} = \oint \rho_2 b |D|^{-3} = 0,$$

as in the noncommutative integral we can use D instead of D_A . As this holds for any b then $\rho_1 = \rho_2 = 0$. Therefore, the only nonvanishing coefficient is ρ_3 and since $\rho_3^2 = 1$ we recover $\Gamma = \sigma^3$ (up to the sign, of course). \square

6.1 Compatible strong connections over \mathbb{T}_θ^3

Next, we turn to the space of possible connections ω . For simplicity, we consider the usual unperturbed Dirac operator D given by (6.1) over \mathbb{T}_θ^3 (i.e. we take $A_i = 0, i = 1, 2, 3$).

Here is the full characterization of the possible connections, according to the definition (5.3):

Lemma 6.6. *A $U(1)$ connection over \mathbb{T}_θ^3 with the $U(1)$ action defined as in (6.2) is a one-form:*

$$\omega = \sigma^3 + \sigma^2 \omega_2 + \sigma^1 \omega_1, \quad (6.4)$$

where $\omega_1, \omega_2 \in \mathbb{T}_\theta^2$ are $U(1)$ -invariant elements of the algebra \mathbb{T}_θ^3 . Every such connection is strong.

Proof. Any $U(1)$ -invariant one-form could be written as $\sum_i \sigma^i \omega_i$, where all ω_i are from \mathbb{T}_θ^2 . Since $\sigma^3 = U_3^{-1}[D, U_3]$, from the condition related to δ we obtain $\omega_3 = 1$. \square

Finally, we give the Dirac operators compatible according to definition 5.9 with an arbitrary selfadjoint connection ω on the noncommutative three-torus.

Lemma 6.7. *For any selfadjoint connection ω the compatible Dirac operator has the form:*

$$D_{(\omega)} = D - (\sigma^2 J \omega_2 J^{-1} + \sigma^1 J \omega_1 J^{-1}) \delta_3. \quad (6.5)$$

The proof follows straight from direct calculation based on the proof of the proposition 5.7. We have an immediate corollary:

Corollary 6.8. *The only connection, compatible with the fully $(U(1))^3$ equivariant Dirac operator (6.1) on the noncommutative three-torus \mathbb{T}_θ^3 is: $\omega = \sigma^3$.*

Finally, observe that the new family of Dirac operators gives a class of new, spectral geometries over the noncommutative torus. Although they are not real, as $D_{(\omega)}$ is not compatible with the real structure, we needed a *real spectral triple* as a background geometry providing us with necessary tools (in particular, the differential calculus).

The properties of the new class of Dirac operators are yet to be investigated, in particular, their spectral properties. However, we expect that they will correspond to some locally non-flat geometries.

7 Conclusions

We have attempted to reconstruct the compatibility conditions between the metric geometry (as given by the Dirac operator) and connections (as defined in the algebraic setup for the Hopf-Galois extensions) on noncommutative $U(1)$ bundles. Although this project has still to be further developed, we have encountered several new and interesting phenomena. First of all, we observe that the existence of real spectral triples is necessary to provide a kind of background geometry. Furthermore, there are many compatibility conditions, which are necessary to impose on the spectral triple, apart from the simple requirement of $U(1)$ -equivariance. We have demonstrated that in the case, where all assumptions are met it is possible to consistently extend the algebraic definition of strong connections to the case of differential calculi given by Dirac operators.

The requirement of compatibility condition between the connection and the metric (given implicitly by the Dirac operator) has led us to the discovery of a new family of Dirac operators. This indicates that these are not objects introduced *ad hoc* but have a deeper geometrical meaning. We postpone the study of their properties to future work.

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